Measuring uncertainty and criticality in network planning by PERT-path technique

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The use of uncertainty and criticality measures in network planning of engineering projects by PERT networks is limited in practice because of their unreliability or complexity. The measures appear as inappropriate to support the definition of contractual obligations when based on uncertain information. The author proposes new uncertainty and criticality measures based on the identification and analysis by PERT-path technique of possible completion sequences of project activities. The measures prove to be effective in project planning and contracting. An algorithm and a stochastic analytical model suitable for the quantification of the measures concerned are proposed. A numerical example is developed. © 1997 Elsevier Science Ltd and IPMA

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Introduction

Failure risks of an engineering project mainly depend upon difficulties inherent to the project, human skills, unpredictable and uncontrollable events and upon uncertainty of information on which decisions are based. A relevant aspect of the last form of risk is the ability of contractual parties to manage at best information available in technical documents supporting contract definition. Project planning is of particular importance in this respect as it contributes to define contractual technical documents.

The drawing up and agreement on planning marked by high uncertainty often leads both contractual parties to commit obligations without full awareness. If the plant is very complex and its implementation subject to great uncertainty, owner and contractor often agree to share failure risks. The goodness of risk appraisal depends on planning estimates reliability and on planning technique adopted. Several ways of conceiving project planning exist and there are as many ways of formalising it. Project modelling results only in a formal and incomplete representation of planner's ideas and of risks he perceives. Yet, once planning has been shared between owner and contractor, project modelling can lead to contract obligations. In this phase uncertainty and criticality measures contribute to estimate project risks. The measures are widely investigated in the literature concerned. They are, however, sometimes unreliable, sometimes difficult to determine, and anyway unsuited at contract stage to estimating project risks. In this scenario new uncertainty and criticality measures in project planning and contracting are needed.

After reviewing main results in the literature concerning uncertainty and criticality in network planning of projects, this paper addresses the theoretical framework to define and calculate new measures for project planning and contracting.

Uncertainty and criticality measures in network planning

Criticality and uncertainty measures concerning the evolution of a project are of varying complexity and reliability according to the network technique adopted in project planning.

One of the most widely used techniques is the Critical Path Method (CPM). CPM assumes that activity times are not subject to uncertainty and bases the identification of project criticality on the slack time allowed for each activity completion. The attention of planners is mostly devoted to zero total slack time activities, that is to those activities belonging to the so called "critical path".

Differently from CPM, PERT assumes that activity time varies randomly according to probability density functions (pdfs) which usually are of Beta type. Under the assumption of statistical independence among activity times, the pdf of project completion date is determined. This pdf is assumed to be normal, with expected value and variance equal to the sum, respectively, of expected values and variances of activity times belonging to the critical
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The pdf determined following this approach leads to estimate the probability that project time exceeds a given value.

Estimates given by PERT are approximate. In particular, they tend to underestimate project time. More accurate criticality and uncertainty measures are provided by PERT Monte Carlo simulation. As activity times are defined in probabilistic terms, the critical path of a project cannot be identified following a deterministic approach. Under those circumstances the “criticality index” of an activity \( A \), \( CI(A) \), and the “project risk”, \( PR \), are defined as:

\[
CI(A) = \text{Prob}(A \text{ Critical path})
\]

\[
PR = \text{Prob}(t > T); \text{ probability that project time, } t, \text{ exceeds a given value, } T.
\]

Although simulation is a powerful and conceptually simple tool, it requires considerable computational efforts. Therefore this approach has to be considered as suited to cases of limited size. A theoretical approach to approximate evaluation of criticality indexes in PERT networks is developed by Dodin and Elmaghraby. After defining ‘path’ as an orderly succession of activities occurring between the starting node and the arrival node of a PERT network, the criticality index of the \( h \)th path, \( CP(n_h) \), belonging to the set \( P \) of possible paths, is defined as the probability that its time, \( T(n_h) \), exceeds that of any other path:

\[
CP(n_h) = \text{Prob}\{T(n_h) \geq T(n_\ast) \} \quad \forall \pi_\ast \in P, \pi_\ast \neq n_h
\]

By \( CP(n_h) \) the criticality index of a generic activity \( A \) is calculated as:

\[
CP(A) = \sum_{A \in n_h} CP(n_h).
\]

New estimators of arc and path criticalities in stochastic activity networks have been recently proposed; they are proved to be more statistically efficient that standard Monte Carlo estimators. Further uncertainty and criticality measures have been proposed with reference to activity networks marked by general precedence relations. These networks, differently from PERT networks, do not impose a strict dependence of the beginning of one (or more) activities from the completion of one (or more) previous activities. The proposed criticality measures are related to the nature of precedence relations between activities.

Complexity of uncertainty and criticality measures grows if complexity of the approach followed increases as well. A survey focusing on convenience domains for PERT-type network techniques (CPM, PERT and VERT) of different complexity is reported by Kidd.

The evolution of PERT-type network techniques has enabled more accurate, though more complex, representations of a PERT system without substantial variations to the original idea of the PERT approach. New representations can be based on deterministic as well as stochastic approaches, with different pdfs and precedence constraints. This evolution was also made possible thanks to the enhanced computer software and hardware tools which support project planning. The most simple techniques in literature allow uncertainty and criticality measures marked by a lower reliability while more accurate techniques, although more reliable, are significantly more complex from the theoretical and/or computational point of view. Hence, measures proposed in literature are scarcely adopted and do not appear as appropriate to support project planning and contracting based on uncertain information.

New theoretical tools are necessary. PERT-path Network Technique (PPNT), a recently proposed approach, contributes in overcoming the above mentioned limitations as it provides the theoretical framework to define and calculate new uncertainty and criticality measures.

Fundamentals of PPNT

The fundamental idea on which PPNT is based is the empirical observation that the actual progress of a project cannot be simply defined on the set of project activities completed at a given date but also on the order and on the completion date of those activities. The completion of an activity according to different completion sequences enables the beginning of other activities, depending on the previous one, at different dates and under different conditions. PPNT, differently from the classic PERT or CPM approaches, enables project planning by analysing project evolutions which take into account all project activities and not only those belonging to the critical path. Therefore the approach highlights the importance of non-critical activities as well in network planning.

PPNT makes use of the same information that is normally available at project planning stage. In particular, given a PERT network and the pdf of the times of the \( N \) project activities, the PPNT allows the analysis of possible evolutions of a PERT system by the state variable:

\[
\pi_k = \{A_{i_1,k}, A_{i_2,k}, \ldots, A_{i_k,k}, 0, \ldots, 0\}, \quad k = [1, N], \quad i_k = [1, n_k]
\]

where \( A_{i,k} \) is the \( k \)th activity completed at the \( i_k \)th activity completion sequence that is defined as “path-state” \( \pi_k \); \( n_k \) is the number of path-states having \( k \) activities completed. The values of the state variable \( \pi_k \) represent generic evolutions, partial (\( k < N \)) or complete (\( k = N \)) of the project. If \( k = 0 \), \( n_0 = n_{1,0} \); PERT system is the initial state where no activity is completed. All paths consistent with logical and time constraints are possible at planning stage. When the project is already under way, at a given date the allowed paths will be only those that the completion sequence followed so far admits as possible. When the project has been completed a single path will have been followed.

If a PERT system is at a given path-state, the completion of an activity enables the transition to a contiguous state. The probability of transition between the two path-states depends upon the stochastic process, time and state dependent, describing the evolution of the PERT system. The transitions graph is defined as PERT-path Network (PPN). PPN has a tree topology with root in \( \pi_{1,0} \) and nodes in the states \( \pi_{i,k}, i_k = 1, n_k, k = 1, N \). PPN topology depends upon PERT network topology and upon definition domain of activity times.

Details about PPNT and an exact algorithm for the determination of PPN topology and variability ranges of activity time consistent with each possible evolution of a PERT system, are reported by Mummolo. In Mummolo and lavagniolo (1994) PPNT is applied to a case study. The results obtained are compared with those that can be determined by the classic PERT approach. It follows that the latter, given the same initial planner’s estimates, has a tendency to underestimate project time. Such a result has already been outlined in the literature by numerical simulation of PERT and by approaches to network planning that consider stochastic variability of all project activities and not only of those activities belonging to the critical path.
Uncertainty and criticality measures in PPNT

Before defining uncertainty and criticality measures in PPNT, some general remarks are necessary. Consider N project activities having deterministc times. The set of logical precedence constraints among activities is defined by a PERT network (PN) and identifies the only activity completion sequence that is consistent with the given times.

On the contrary, in case of activities subject to stochastic variability, there are several completion sequences that are consistent with PN topology and with variability domains of activity times.

The number of possible completion sequences and related occurrence probabilities are measures of uncertainty related to the evolution of a project.

Number of completion sequences of a project

The number of possible evolutions of a project theoretically ranges between 1 and N!. The lower limit is reached when, irrespective of PN topology, activity times are considered as deterministically known. The upper limit is reached in case of concurrent activities with times (random variables) defined in upper boundless domains. In practice the number of possible evolutions of a project ranges between the aforementioned limits and tends to be very high. This highlights the inherent uncertainty at planning stage in assessing the project completion path. An exact algorithm highlights the inherent uncertainty at planning stage in assessing the project completion path. An exact algorithm to calculate the number of possible completion sequences of a project having a given PN topology and each pdf defined in an upper boundless domain, is proposed in the appendix of this paper. The number depends upon PN topology only. It certainly is an upper limit for the actual number of paths. This last can be determined once the finite domain of the pdf of each activity time has been defined. Completion paths of a project tend to increase if both number of concurrent activities and size of activity times domains increase as well.

Transition and path probability

Probability of each completion sequence depends upon transition probability between two PPN contiguous path-states. Each transition probability depends upon the nature of the stochastic process marking a project evolution. In case of maximum uncertainty, that is all activity times are exponentially distributed, the stochastic process is a continuous-time discrete-state Markov process. This process is marked by a complete lack of "memory", that is the process is "time and state independent". In this case the transition probability between a couple of contiguous states in a PPN only depends on the starting state of transition and not on the path followed or on the dates when transitions have occurred.

If times of some activities are assumed as being distributed according to pdfs different from the exponential distribution, the process has "memory" of the path followed. Transition probability is therefore influenced by the path followed and by the date when each transition has occurred.

Transition probability and entropy

Consider a PERT system is at the pth path-state, τ_p,k, among those having k activities completed. The completion of an activity, say A_i, enables the transition from τ_p,k to a contiguous state, τ_{i,k+1}. Let Ω_i be the set of activities that can be started at τ_p,k and activities already started, but not yet completed in that state. If an activity of Ω_i starts at τ_p,k then the pdf of the related time is that initially estimated by planners. If an activity of Ω_i starts in a path state, τ_{r,k-m}, m ≥ 1, which precedes τ_p,k but is not yet completed in that state, then the related pdf differs from the initial one. As far as this last activity type is concerned, the residual time of A_i, d_i(A_i), can be calculated as:

\[ d_i(A_i) = D(A_i) - (\pi_{p,k} - \pi_{r,k-m}) \]  

where

\[ D(A_i) \] is the initial time of A_i, \( \pi_{p,k} \) is the arrival time of PERT system in the current path-state, \( \pi_{r,k-m} \) is the arrival time of PERT system in \( \pi_{r,k-m} \). As \( D(A_i) \) and \( (\pi_{p,k} - \pi_{r,k-m}) \) are random variables (rvs), consequently \( d_i(A_i) \) is also a rv. Assuming that total or residual times of Ω_i activities are independent rvs, the transition probability from τ_p,k to a contiguous state τ_{s,k+1} can be calculated as:

\[ \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} = \text{Prob}\{\pi_{s,k+1}\} \]  

Equation 2

where \( f_\pi \) is the pdf or residual time of activity \( A_i \), enabling the transition from \( \pi_{p,k} \) to \( \pi_{s,k+1} \), and \( R_q \) is the reliability function of \( A_i \) time, \( \forall A_i \in \Omega_i, A_s, A_q \neq A_i \).

In the case of exponentially distributed activity times, the memorylessness property of the stochastic process describing project evolution leads to simplify Equation (2) as:

\[ \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} = \frac{1}{E[t(A_p)]} \]  

where \( E[t(A_p)] \) is the expected time of activity \( A_p \).

Equation 2 (or (2')) provides an uncertainty measure concerning the evolution of a PERT system between two contiguous states of PPN.

An uncertainty measure concerning all possible transitions from \( \pi_{p,k} \) to its contiguous path-states in a PPN can be defined by the "entropy by Shannon" of transitions:

\[ H(\pi_{p,k}) = - \sum_{A_i \in \Omega_i} \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} \cdot \log(\text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\}) \]  

where \( \sum_{A_i \in \Omega_i} \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} = 1 \).

The rationale for using the "entropy" estimator is the mathematical theory of communications. However, the axiomatic properties of the estimator make it widely adopted in several fields of knowledge (e.g. engineering, ecology, statistics, etc.) to a measure to quantify uncertainty inherent to a probabilistic phenomenon is required.

If one transition is allowed from \( \pi_{p,k} \) to its contiguous path-states in a PPN, then entropy gets its minimum value, \( H_{\text{min}}(\pi_{p,k}) = 0 \), highlighting the absence of uncertainty in transition starting from \( \pi_{p,k} \). If \( n_{\pi_{p,k}} > 1 \) and a uniform distribution among transition probabilities is assumed (\( \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} = \frac{1}{n_{\pi_{p,k}}} \)) the entropy gets its maximum value, \( H_{\text{max}}(\pi_{p,k}) = \log(n_{\pi_{p,k}}) \). Entropy ratio \( r(\pi_{p,k}) = H(\pi_{p,k})/H_{\text{max}}(\pi_{p,k}) \) gets values ranging from 0 (minimum uncertainty) to 1 (maximum uncertainty).

Transition entropy ratio \( r(\pi_{p,k}) \) and \( \text{Prob}\{\pi_{p,k}, \pi_{s,k+1}\} \) are objective indexes to detect more uncertain transitions and to identify causes which are responsible for higher
uncertainty to complete a project according to an agreed evolution. This could lead to change initial planning estimates in order to agree on new and more reliable contractual obligations.

**Path probability and entropy**

Once the probability of each transition of a PPN has been calculated by (2) (or (2'), the probability of each path (partial or complete) can be evaluated as:

\[ \text{Prob}\{\pi_{n+1} = \pi_{n, k+1}\} = \text{Prob}\{\pi_{n, k}\} - \text{Prob}\{\pi_{n, k}\} \cdot \text{Prob}\{\pi_{n, k}\} \]  (3)

with the initial condition: \( \text{Prob}\{\pi_0 = \pi_{1, 0}\} = 1. \)

Generally speaking, although the initial number of possible paths is high only a few among them represent the most probable project evolutions. Uncertainty inherent to possible project paths can be determined by setting \( k+1 = N \) in Equation (3) and calculating the project entropy estimator defined as:

\[ H = - \sum_{n=1}^{n_N} \text{Prob}\{\pi_N = \pi_{n, N}\} \cdot \log(\text{Prob}\{\pi_N = \pi_{n, N}\}) \]

where \( \sum_{n=1}^{n_N} \text{Prob}\{\pi_N = \pi_{n, N}\} = 1 \) and \( n_N \) is the number of complete project paths.

This parameter ranges between a minimum value, \( H_{\text{min}} = 0 \), in the case of a project with a single possible completion sequence, and a maximum value, \( H_{\text{max}} = \log(n_N) \), in the case of complete sequences having equal occurrence probability (1/\( n_N \)). Project entropy ratio \( \sigma = H/H_{\text{max}} \) taking values ranging between 0 and 1, is a measure of uncertainty concerning project evolutions.

**Path completion date**

A further measure of uncertainty is the risk that project time exceeds a given value. Differently from PERT approach, this measure can be determined by PPNT with reference to each possible project completion sequence. The pdf \( f(t(\pi_{n, N})) \), \( t_{1, N} \), of the project time calculated according to \( \pi_{n, N} \) can be determined by convolution of pdfs of time intervals between contiguous transitions belonging to the path concerned, as

\[ f(t(\pi_{n, N})) = f_1(t) \cdot f_{2, n_1, n_2} \cdot f_{3, n_2, n_3} \cdots f_{k, n_{k-1}, n_k} \]  (4)

the generic term of convolution, \( f_{n_{k-1}, n_k} \), can be determined as:

\[ f_{n_{k-1}, n_k}(t) = \frac{\int_{-\infty}^{\infty} f_1(\tau) \cdot \prod_{n_{k-1} \neq n_{k-1}, \tau} R_\tau(t) \cdot d\tau}{\int_{-\infty}^{\infty} f_1(\tau) \cdot \prod_{n_{k-1} \neq n_{k-1}, \tau} R_\tau(t) \cdot d\tau} \]  (5)

Function \( f_1 \) (and consequently \( R_\tau \) as well) appearing in (2) and (5) can be determined by the method of the auxiliary variable,\(^{11}\) considering activity times (total or residual) as independent \( v_\tau \).

The method will be first applied under the assumption that the time interval elapsed between the starting date of activity \( A_1 \) and the arrival date of PERT system in the current path-state \( \pi_{n, k} \) is formed by the time of one activity, \( A_1 \), started in \( \pi_{n, k} \) (\( m = 1 \)). In this case, setting \( D(A) = v_\tau \cdot \tau_{n, k} \) in Equation (1), residual duration of \( A_1 \) is:

\[ d_r(A_1) = D(A_1) \cdot D(A_2) \]  (6)

Setting:

\[ z = D(A_1) \]

the pdf of \( d_r(A_1) \) can be obtained as:

\[ f_s(d_r(A_1)) = \frac{\int_{-\infty}^{\infty} f_s(z) \cdot f_s(d_r(A_2) + z) \cdot dz}{|J[D'(A_1), D'(A_2)]|} \]

where \( f_s \) and \( f_s' \) are pdfs of \( A_1 \) and \( A_2 \), respectively, as initially estimated by planners, and \( |D'(A_1), D'(A_2)| \) is the absolute value of the Jacobian of the system Equations (6) and (7), calculated in the roots: \( [D'(A_1), D'(A_2) = \pm (A_2 \pm A_1)] \).

On the other hand, if the time interval elapsed between the starting date of \( A_1 \) and the arrival date in \( \pi_{n, k} \) is formed by more than one activity (\( m > 1 \), then residual duration of \( A_1 \) is:

\[ d_r(A_1) = D(A_1) \cdot \sum_{m=1}^{m} D(A_1), \quad m > 1 \]  (8)

The auxiliary variable method can still be applied considering in sequence, one at a time, the terms of the summation in Equation (8):

\[ d_r(A_1) = D(A_1) \cdot D(A_2) \cdot D(A_3) \cdot \cdots \cdot D(A_m) = d_r^{(m)}(A_1) \cdot D(A_m) \]

The model allows the calculation of \( f(t(\pi_{n, N})) \) and consequently, risk that project time evaluated according to a given \( \pi_{n, N} \) exceeds a target value.

**Path ‘utility’ and project ‘utility’**

Generally speaking, different technical and/or economic objectives can be pursued in an engineering project, according to different activity completion sequences. For example, different time sequences of project cash flows can determine different net present values, that is a different ‘utility’. Moreover, project time varies if project evolution (path) varies as well; consequently, penalties for a delay in the delivery date depends on path followed and affects ‘utility’ of that path. Each path has an occurrence probability to occur. It makes sense to define the expected project ‘utility’ as:

\[ U = \sum_{n=1}^{n_N} u(\pi_{n, N}) \cdot \text{Prob}\{\pi_{n, N}\} \]

where \( u(\pi_{n, N}) \) indicates the ‘utility’ related to the \( n \)th complete path of a project.

Ratio \( \eta(\pi_{n, N}) = u(\pi_{n, N}) \cdot \text{Prob}\{\pi_{n, N}\}/U \) is defined as a ‘utility index’ of the \( n \)th complete path. \( \eta(\pi_{n, N}) \) varies within range [0,1]. It tends to 0 with the increase in number of paths; the ratio is equal to 1 if the project has a single path. If probability and/or ‘utility’ of \( \pi_{n, N} \) increase then \( \eta(\pi_{n, N}) \) increases as well. Paths which have lower ‘utility index’ are more crucial since \( \eta(\pi_{n, N}) \) expresses the attitude of path \( \pi_{n, N} \) to contribute to project ‘utility’. Management of both contractual parties should pay attention on whether agreed planners’ estimates, considered in a deterministic approach to project planning (e.g. CPM), force the project to follow a critical evolution, that is a path with a low ‘utility’ and occurrence probability. If this is the case new planners’ estimates should be provided.

A numerical example

In the following a numerical example is developed. The example considers a few project activities subject to
Table 1 Data of the numerical example: precedence constraints and
eventual value of each activity time

<table>
<thead>
<tr>
<th>Activity No.</th>
<th>Expected time [time unit]</th>
<th>Predecessors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3,4</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

precedence constraints reported in Table 1. Table 1 also
reports the expected value of each activity time, supposed
distributed according to an erlang pdf of order K. Two
different cases will be considered in the example, assuming
K=1 and K=2. In the first instance the pdf of each activity
time degenerates in an exponential distribution; the
stochastic process describing project evolution is marked
by lack of memory. In the case of K=2, the process has a
memory of path transitions followed and of dates when they occurred.

Figure 1 shows the PPN corresponding to the PN defined
by the precedence constraints reported in Table 1. For the
sake of simplicity path-states have been numbered in
sequential order rather than by the symbolism adopted in
the theoretical model. Although the example is very simple,
the network is composed of 91 path-states, 26 of which
(from N.65 to N.90) are complete project evolutions
consistent with logic precedence constraints. The number
of path-states and PPN topology are calculated by the exact
algorithm reported in the appendix of this paper.

Figures 2 and 3 show the probability values of complete
paths, calculated by the analytical model proposed, for
cases of K=1 and K=2, respectively. The figures also
report the values of path probabilities obtained by Monte
Carlo simulation with a random generation of 10,000
simulation runs for each case. The comparison between the
values obtained by the two different approaches highlights
the reliability of the analytical model, since it provides
exact results in the case of K=1 and well approximated
results in the case of K=2. Approximations are due to the

Figure 1 Numerical example: PERT-path network of all project paths (100% probability level)
Figure 2  Numerical example: final paths probability in case of erlang (K=1) activity time distributions; comparison between the results obtained by the analytical model (bold bars) and those obtained by numerical simulation.

hypothesis of statistical independence of total and residual activity times. This hypothesis is strictly verified in the case of $K=1$. On the contrary, it is less strictly so in case of activity time pdfs different from an exponential distribution. In this last case, however, it can still be considered as acceptable. Actually, during initial phases of a project plan, the low number of partial paths (see PPN tree topology) tends to increase their occurrence probability. During these
phases, evaluation error of probabilities is generally small as the effects of stochastic dependency among activities, that have just or recently been started, are not yet evident. Time pdf of those activities do not substantially differ from those estimated by planners. On the contrary, the completion of activities occurring towards the end of a project is much more influenced by time elapsed and the effect of stochastic dependency on the path followed so far can be evident. The proliferation of paths reduces their occurrence probability to a considerable extent and increase evaluation errors. However, even considerable evaluation errors concerning very low occurrence probability paths are of scarcely any importance for decision makers.

The maximum error in estimating the probability of the three most probable paths \((\pi_{87}, \pi_{90}, \pi_{86})\) is in the worst case \((K = 2)\) lower than 3%. The cumulated probability for all complete paths (case: \(K = 2\)) ordered by decreasing probability, is shown in Figure 4. By eliminating from PPN of Figure 1 the 23 complete paths which have occurrence probability lower than 6%, the PPN of Figure 5 is obtained. The new graph is much simpler than the complete one and makes it possible to focus planners’ attention on the three most probable paths. These ones represent the set of possible evolutions that the project will follow with a total probability of about 65%. The most probable path is \(\pi_{87}\). It has a 32.5% probability to occur. This path corresponds to the path that would be obtained if the values of activity times were stated as deterministically known considering them equal to the corresponding expected values. The low value of \(\pi_{87}\) probability is a clear example of how planners’ uncertainty makes project evolution all but deterministically identifiable. The considerable degree of uncertainty concerning possible project evolutions is moreover underlined by the considerable value taken by project entropy ratio, \(r = H/H_{\text{max}} = 0.657\). Assuming for the sake of simplicity that the complete paths have equal ‘utility’, the obvious result is that the ‘utility’ of the project is equal to that of each path and that \(\eta(\pi_{6, N}) = \text{Prob}(\pi_{6, N})\). Hence most critical paths are also those ones having the lowest occurrence probability. In this case, management of both contractual parties could be interested in increasing occurrence probability of \(\pi_{87}\) thereby reducing uncertainty of project evolution. Detecting the sources of uncertainty is enabled by analysing the values of transitions’ probability (in round brackets) and transitions’ entropy ratio reported in Figure 5. The latter values outline that the most uncertain transition occurs when the PERT system is the path-state \(n.2\): in this state, transition entropy ratio gets its maximum value, \(r(2) = 0.910\). This stems from both the number of concurrent activities \((A_1, A_4\) and \(A_6)\) and from the related distribution of transition probabilities \((0.157, 0.512\) and \(0.331)\). Reduction in entropy ratio \(r(2)\) can be achieved by increasing the value of transition probability \(\text{Prob}(2,6) = 0.512\) thereby allowing the PERT system to evolve according to \(\pi_{87}\) with a higher occurrence probability. Planners can pursue such an objective by varying expected activity time of the concurrent activities (e.g. reducing \(A_4\) expected time). Of course, new activity times require a new resources allocation among activities. A similar line of reasoning can be performed to make \(\pi_{87}\) far more probable by analysing the PERT system at the path-state \(n.6\) which has the second highest transition entropy ratio \((r(6) = 0.834)\).

The zero of the project completion date referred to the most probable path, \(\pi_{87}\), has been calculated by Equations (2) and (3) in case of time pdfs distributed according to erlang with \(K = 2\). The expected value (expressed in time units, t.u.) of the distribution is equal to \(E[t] = 27.11\) (t.u.). According to PERT, the project completion date

![Figure 4](image-url)  
**Figure 4** Numerical example: cumulative probability of complete paths ordered according to decreasing probabilities, in the case of erlang \((K=2)\) activity time distributions
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Figure 5 Numerical example: PERT-path network at 65% probability level, transitions' probability (in round brackets), transitions' entropy ratio, and paths' probability in case of erlang (K=2) activity time distributions

estimated on the critical path (activities 1-3-5) is normally distributed with parameters $\mu = 23$ (t.u.) and $\sigma = 9.16$ (t.u.). The comparison between the expected values shows that PERT can lead to underestimating project time by over 15% as compared to the expected time of the most probable path. Finally, in Figure 6 are compared project risk ($\text{Prob}(T_{r} > T)$) values related to the most probable path $\pi_{87}$ (PPNT) and to the critical path (PERT). Both PPNT and PERT are stochastic approaches. However, the latter underestimates project risk in the comparison considered as it refers to the critical path which neglects the stochastic influence of non-critical activities on project time.

Managing planners' uncertainty

The numerical example shows how risky can it be to refer to a deterministic planning during the contractual phase whenever initial estimates are, as is often the case, subject to uncertainty. A planning considering everything as certain leads to a single evolution mode for the project and hides risks related to planners' uncertainty. Contract specifications should, in this framework, be defined with reference to possible project evolutions and related occurrence probabilities. In particular, planning by PPNT can lead to the identification of:

Figure 6 Numerical example: project risk according to the critical path (PERT) and to the most probable path, $\pi_{87}$ (PPNT), in the case of erlang (K=2) activity time distributions
—a very high number of paths with a low occurrence;  
—several paths with a medium occurrence probability that  
point of view of their 'utility' as well as of their technical,  
economical, logistical and organisational feasibility;  
—a very high number of paths with a low occurrence  
probability of project evolution having a scarce contractual  
relevance.

Each completion sequence of a project relates to a 'utility' that can be estimated in probabilistic terms. Paths, therefore, take different criticality levels in relation to their 'utility' and occurrence probability. If one of higher critical paths is deterministically agreed upon in contractual phase then new planners' estimates are needed in order to maximise the 'utility index' of that path. Detection of sources of uncertainty can be carried out by analysing entropy and probability of each transition belonging to the path concerned. Transitions with higher entropy ratio are the most uncertain. Transitions with lower probability contribute to make a project path less probable. Reduction in entropy as well as increase in transition probability can be obtained by modifying initial activity time estimates that is by a new allocation of resources.

Since the set of these evaluations become objective if planners' estimates are accepted, they can influence contractual obligations to a significant extent. Two major considerations should support project contracting. The first one is the strict relation between planners' estimates in project planning and occurrence probability of project evolutions. The second one is the opportunity for contractual parties to refer to a well defined project evolution as project performances (e.g. project time, 'utility' and risk) can significantly change according to the path followed in project completion.

Summary and conclusions

Available approaches to network planning are either too simple but scarcely reliable, or too complex to be adequately reliable. Such circumstances significantly hamper the practical use of uncertainty and criticality measures that these approaches enable to quantify. The measures appear as unsuited for decision-making process that supports the definition of contractual specifications since they are not related to well identified and agreed project evolutions.

This paper defines new uncertainty and criticality measures related to project evolutions. The measures refer to the recently proposed PERT-path network technique. The technique makes use of the same information normally available for project planning (i.e. precedence constraints and pdfs of activity times), and, differently from classical PERT-type techniques, it considers all possible completion sequences of a project. The number and occurrence probabilities of completion sequences appear as effective uncertainty measures in network planning. These measures are determined in the paper by an algorithm and by a stochastic analytical model.

The concept of path criticality differs from those so far proposed in literature as it is considered in relation to 'utility' and probability of paths along which a PERT system can evolve. The concept of 'utility' can refer to technical and economic performances of a project evolution. The numerical example stresses how risky and non-significant can it be to refer to a classical PERT-type network technique when uncertain information is available. This suggests to resort to new reference paradigms to be adopted in contractual agreements specifications. In particular planners' uncertainty, expressed by probabilistic definition of possible completion sequences of a project, should be quantified and agreed upon by contractual parties as an objective framework to be considered when taking specific commitments.

Planning techniques are only instruments that give a representation of reality as thought of by planner. Events marking "a posteriori" actual project evolution could not allow to comply with forecasts made. This is the limitation of any planning tool and, therefore, of PPNT as well. This last, however, enables a more accurate and objective management of planners' uncertainty than classical network techniques as it proposes a common ground for contractual parties which should take decisions on the basis of all project evolutions consistent with planners' estimates.

Further research should be finalised to propose new contract specifications based on probabilistic evolutions and related performances of a project.

References

9 Mummolo, G. and Javagnilo, R., Managing planners’ uncertainties by the PERT-path technique: an Italian case study INTERNET94 12th World congress, Oslo (June 1994).
Appendix: Calculations of the path-states number of a PERT-path network.

An algorithm that is able to determine the number of path-states corresponding to a given PERT (PN) network topology is proposed in this part of the paper. This number is an upper limit for the actual number under the hypothesis of upper boundless variability domain of each activity time. By the following, the basic steps of the proposed algorithm are reported. For the sake of simplicity, the algorithm will be illustrated with reference to the numerical example developed in this paper. Table A1 reports the results obtained.

Symbolism and definitions

PERT-State Network (PSN) = dual directed graph of a PN. In a PSN each node represents a possible state of the PERT system, that is it defines the set of activities that may be completed at a given date, irrespective of the path followed. $K_{j,t}$ = $j$th state having $t$ activities completed in the PSN $I_t$ = set of $K_{j,t}$ states

$K_{j,t}$ = $j$th preceding state, contiguous to $K_{j,t}$, having $(t-1)$ activities completed
$K_{j,t-1}$ = set of the states $K_{j,t-1}$.

$d_{i}(K_{j,t})$ = number of arcs entering the state $K_{j,t}$ in the PSN $d_{o}(K_{j,t})$ = number of arcs exiting the state $K_{j,t}$ in the PSN $d_{i}^{*}(K_{j,t})$ = equivalent number of arcs entering the state $K_{j,t}$ in the PSN

Step 0: The first step of the algorithm is the PSN determination.

Figure A1 shows the PSN corresponding to the PN defined by precedence constraints reported in Table 1.

Step 1: Calculate $d_{i}(K_{j,t})$ and $d_{o}(K_{j,t})$, $\forall K_{j,t} \in I_t$, $t=0,N-1$. For $t=0$, $d_{i}(K_{1,0})=1$.

Step 2: $t=0$.

Step 3: Calculate the equivalent number of arcs entering $K_{j,t}$ as:

$$d_{i}^{*}(K_{j,t})=d_{i}(K_{j,t})+\sum_{K_{j,t-1}\in I_{t-1}}g(K_{j,t-1})$$

with the initial condition:

Table A1 Numerical example: application of the algorithm to the calculation of path-states

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<tr>
<th>$t$</th>
<th>$K_{j,t}$</th>
<th>$d_{i}(K_{j,t})$</th>
<th>$d_{o}(K_{j,t})$</th>
<th>$\Sigma g(K_{j,t-1})$</th>
<th>$d_{i}^{*}(K_{j,t})$</th>
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Figure A1 ‘PERT State’ Network of the numerical example
$g(K_{1,0}) = 0$. $K_{j,t}$ can be reached from the initial state $K_{1,0}$ by $d_*(K_{j,t})$ preceding paths in the PSN.

**Step 4:** Calculate the number of path-states at the $n$th transition as: $n_t = \sum_{K_{j,t} \in \mathcal{H}} d_*(K_{j,t}) d_0(K_{j,t})$

**Step 5:** if $t < N-1$ then $g(K_{j,t}) = d_*(K_{j,t})-1$ and $t = t+1$: goto Step 3 otherwise goto Step 6.

**Step 6:** Calculate the number of path-states of the PPN associated to the starting PN as $n = \sum_{t=0}^{N-1} n_t$. This calculated value of $n$ does not consider the initial state of the PPN ($\pi_{1,0}$).

Compare the values of $n_t$ calculated by the algorithm (*Table A1*) and the PPN in *Figure 1.*